TRANSPORT COEFFICIENTS OF FULLY IONIZED PLASMAS

By

R. K. PATNAIK

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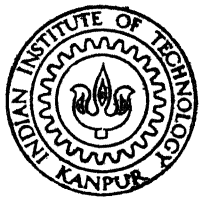

NUCLEAR ENGINEERING & TECHNOLOGY PROGRAMME

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TRANSPORT COEFFICIENTS OF FULLY IONIZED PLASMAS

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of
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By

R. K. PATNAIK

to the
NUCLEAR ENGINEERING & TECHNOLOGY PROGRAMME
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CERTIFICATE

This is to certify that this work on "Transport Coefficients of Fully Ionized Plasmes" has been carried out under my supervision and it has not been submitted elsewhere for a degree.

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October, 1981

In the memory of my father

and

To the enduringly affectionate mother,

with all the modesty and humbleness.

Acknowledgements

Whenever I am let to loose myself in the pages of this work, I cannot help recollecting my days of association with Dr. M.S. Kalra, who was my thesis supervisor and teacher as well. I owe my indebtness to him for exposing me to Transport Phenomena in Plasmas and later encouraging to take up my thesis topic in the same field. From the very inception, this work wound its way to completion under his constructive and insightful nourishment.

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Patnaik Radha K.

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NOMENCLATURE

D	Diffusion constant or Coefficient of diffusivity
e	Electronic charge
f	Velocity distribution function
F or F_1	One-particle distribution function
F_N	N-particle distribution function
G	Correlation function
H_N	Hamiltonian of the N-particles
$\vec{J}$	Heat flux
K	Coefficient of thermal conductivity
k	Boltzmann constant
m	Particle mass
m_e	Mass of electron
m_i	Mass of ion
n_e	Number density of electrons
n_i	Number density of ions
n	Total number density of particles
$\vec{p}$	Momentum
$\vec{p}'$	Dimensionless momentum
$\overline{\overline{P}}$	Stress tensor
$\vec{q}$	Coordinate vector
$r=r_{12}$	Interparticle distance
r_D	Debye shielding length

T	Temperature
$\vec{u}$	Macroscopic velocity
$\vec{U}$	Peculiar or random velocity
$\vec{v}$	Particle velocity
V	Volume
Ze	Ion charge
θ	Temperature (= kT)
ϕ_{12}	Interparticle potential
ε	Plasma parameter
ε_o	Permitivity of free space
λ_e	90^o deflection path for electrons
λ_i	90^o deflection path for ions
$\ell n \Lambda$	Coulomb logarithm
$\vec{\Gamma}$	Particle flux
μ	Coefficient of viscosity

Abstract

Transport coefficients, namely thermal conductivity, viscosity and diffusion constant, of external field-free fully ionized plasmas are examined in the framework of a kinetic description based on BBGKY-hierarchy of equations.

Due to long-range nature of the Coulomb force, leading to multiple collisions and interparticle-correlations, a closed form kinetic description of plasmas can only be obtained by making a series of simplifying assumptions. Several such schemes reported in literature are reviewed.

It is found that the correct functional dependence of plasma transport coefficients on plasma properties, viz. temperature, density, charge and mass of the constituents, can be obtained by making certain very simple assumptions regarding the single particle and two-particle distribution functions, within the framework of hydrodynamic approximation. Absolute numerical values can then be calculated by comparison with measurements or more advanced theory.

Chapter - 1

INTRODUCTION, BACKGROUND AND OUTLINE

1.1 Introduction

A plasma is a quasineutral collection of positively and negatively charged particles. In addition to charged particles, neutral particles may also be present, in which case the plasma is called partially ionized. If no neutral particles are present the plasma is called fully ionized. The transport processes in a fully ionized plasma are the subject of study in the present work.

For a fully ionized plasma, the ionization-recombination behaviour and collision with neutral atoms need not be modelled. In our work we will also ignore the relativistic and gravitational effects, and take plasma free from external electric and magnetic fields.

Plasma behaviour can be studied by employing one of the following models :

1. Individual particle model,
2. Kinetic model,

and 3. Fluid model.

Even with the availability of extremely fast computers, the individual particle model can not be used in almost all cases of practical interest because an astoundingly large number of equations are involved in order to study the system. Thus one has to rely on the kinetic or fluid models.

In the fluid or hydrodynamic model, macroscopic conservation equations for mass, momentum, energy etc are derived phenomenologically without any reference to the macroscopic distribution function. The resulting equations, however, can not be solved without making additional assumptions. The most important of these assumptions is the introduction of "linear rate equations" into the hydrodynamic equations. A potential gradient is associated with the flux or net flow rate of a property based on experimental evidence and experience, and a linear relationship between the flux and the potential gradient is assumed. The transfer process takes place in the direction of decreasing potential. These phenomenological rate equations are : Fick's equation of diffusion, Fourier's equation of heat conduction, Newton's equation of viscosity and Ohm's equation of electrical conduction.

This method, however, has one serious limitation. The coefficients in the rate equations, namely, the diffusion coefficient, thermal conductivity, the coefficient of viscosity and the electrical conductivity, are to be determined purely by experiment. That is, this approach does not provide any theoretical values of the transport coefficients, nor does it give the functional relationship between the transport coefficients and other macroscopic properties of the system.

In the Kinetic model, however, one derives the hydrodynamic equations by taking appropriately weighted moments of the kinetic equation governing the microscopic behaviour of the system. When the kinetic equation is solved in the hydrodynamic approximation, theoretical values of the transport coefficients can be obtained. Also rate equations do not have to be introduced adhoc, as in the phenomenological method.

The information obtained about the transport coefficients from the kinetic theory, however, depends on the assumptions made in solving the kinetic equation. It is shown in this work that for small deviations from equilibrium the functional dependence of the transport coefficients of fully ionized plasmas can be deduced in a relatively simple manner. Absolute numerical values can then be calculated by comparison with measurements or more advanced theory.

1.2 *Background and Literature Review*

The mathematical theory of thermal conduction, viscosity and diffusion in neutral gases till recently has been based almost entirely on Boltzmann kinetic equation [1]. The equation is applicable only to dilute neutral gases because of the assumptions made in deriving the equation, namely,

(a) Interparticle forces are short-range. A consequence of this is that the mean free path between collisions is much larger than the range of interparticle forces, or the mean free time is much larger than the collision time.

(b) The collisions are only binary.

(c) The particles are statistically independent.

Several methods for solving the Boltzmann equation have been developed, for example those of Chapman -Enskog, Burnett and Grad [2]. Enskog has also applied Boltzmann equation to dense gases. The Boltzmann equation is, however, completely unsuitable for plasmas, because none of the above mentioned assumptions are valid due to the long range nature of the Coulomb force. In particular, the following effects have to be explicitly accounted for in the case of plasmas:

(a) The particles are no longer statistically independent. In other words they are statistically correlated.

(b) All collision in reality are multiple collisions. In fact, in some treatments of plasmas, for example, Fokker-Planck theory, short range binary collisions can be ignored.

(c) No well-defined mean free path or time exists for plasmas because the particles are interacting all the time.

Bogoliubov [3] has presented a new approach to the formulation of kinetic theory based on Liouville equation [4] for the N-particle distribution function. By defining the reduced s-particle distribution functions, it is shown that the Liouville equation can be put in the form of the completely equivalent system of equations, called BBGKY hierarchy of equations [5,6]. It will be discussed in detail in Chapter 2. By making certain assumptions about the time evolution of the

distribution functions and about the nature of the correlations in the past, Bogoliubov has shown that, as a first approximation, the Boltzmann equation can be derived from the BBGKY equations. Choh and Uhlenbeck [2] have solved the BBGKY equations in higher order approximations, thus accounting for the multiple collisions and correlation effects for dense gases.

Frieman and Sandri [8,2,10] have also developed a theory to solve the BBGKY equations with somewhat less restrictive assumptions, particularly, about the correlations in the past. Rostoker, Rosenbluth and Ischikawa have developed another approach for truncating the BBGKY hierarchy of equations [11,12, 13]. This is based on the concept of cluster expansion well-known in statistical physics [14,15]. Wu [2] has shown that the essential features of all these approaches are equivalent, eventhough, the mathematical steps involved are quite different.

Guernsey and Lenard [2,17] have extended Bogoliubov's theory for neutral gases to plasmas. The essential difference involved is in the expansion parameter used. For the case of neutral gases the density of gas is used as an expansion parameter. For plasmas, it is the ratio of potential energy to kinetic energy, called the plasma parameter, which is used for expansion. This will be discussed in more detail in Chapter 2. For small values of plasma parameter, these authors have worked out the equilibrium one-particle and two-particle distribution functions. These will be utilized in our work in Chapter 4.

Sundaresan, Wu and Rosenburg [18,19,20] have applied Bogoliubov's theory to inhomogeneous plasmas and derived an expression for thermal conductivity of fully ionized plasmas for small temperature gradients. The dominant term in their expression [20] agrees well with the expression derived earlier by Spitzer and Harm [21,22] on the basis of Fokker-Planck equation.

In this work we have derived expressions for thermal conductivity, viscosity and diffusion coefficient of a fully ionized plasma by utilizing one-particle and two-particle equilibrium distribution functions as obtained by other authors [2,17]. Small deviations about the equilibrium solutions are assumed due to small gradients in temperature, density and macroscopic or hydrodynamic velocity. The weighted integrals of the kinetic equation are obtained yielding appropriate expressions for the transport coefficients. The mathematics involved in this approach is considerably simpler than that needed in other approaches.

1.3 *Outline of the Present Work*

In Chapter 2, a detailed description of the kinetic theories of gases and particularly plasmas is given. The emphasis is on BBGKY approach. The equilibrium form of the one-particle and two-particle distribution functions is presented and discussed. In Chapter 3, the hydrodynamic equations are derived by taking the weighted moments of the

kinetic equation, and the transport coefficients are identified. In Chapter 4, the expressions for the transport coefficients are derived based on simplifying assumptions mentioned above. Chapter 5 presents a summary of this work, reiterating the main conclusions and suggesting areas that need further work.

Chapter - 2

KINETIC DESCRIPTION OF NEUTRAL GASES AND PLASMAS

2.1 Maxwell-Boltzmann Kinetic Theory

The kinetic theory of neutral gases, as it is presently understood, is based on the work of Maxwell and Boltzmann. The assumptions made by all previous workers, namely, that all molecules of a gas move with the same speed, was abandoned and the random character of molecular motion was recognized. The Maxwellian distribution of velocities for a uniform gas in equilibrium, and the law of equipartition of the mean particle energy in a mixture of gases were firmly established [2,7]. In an attempt to give a rigorous justification of Maxwell's assumptions, Boltzmann derived an integro-differential equation to describe the evolution of the velocity distribution in space and time. This integro-differential equation is now known as Boltzmann equation.

2.1.1 The Velocity-Distribution Function

A gas particle can be characterized by its position $\vec{r}$ in space and its velocity $\vec{v}$ or momentum $\vec{p}$. We will be using $\vec{v}$ or $\vec{p}$ according to convenience. Information is not only required about the spatial distribution of particles, but about their velocity and momentum distribution as well. Only with the latter we can study the momentum and energy flows.

The simplest function that contains the desired information is the velocity distribution function, $f(\vec{r}, \vec{v}, t)$, defined in such a way that $f(\vec{r}, \vec{v}, t)\, d\vec{r}d\vec{v}$ is the expected number of particles, at time t, in a volume element $d\vec{r}$ located at $\vec{r}$, whose velocities lie in $d\vec{v}$ about the velocity $\vec{v}$. The volume elements $d\vec{r}$ and $d\vec{v}$ are, by definition,

$$d\vec{r} \equiv d^3r \equiv dxdydz, \tag{2.1}$$

and

$$d\vec{v} \equiv d^3v \equiv dv_x dv_y dv_z \tag{2.2}$$

where x, y and z represent the cartesian coordinates and v_x, v_y and v_z are components of $\vec{v}$ in directions of x, y and z, respectively. The macroscopic or average properties of the gas as well as the flux vectors of mass, momentum and energy, can be defined in terms of the velocity distribution function (see Chapter 3).

2.1.2 The Boltzmann Equation

Boltzmann derived the kinetic equation governing the evolution of the velocity distribution function in space and time. Many modern derivations have been developed by several authors [2, 7, 1]. The equation for $f(\vec{r}, \vec{v}, t)$ can be written in the form

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_c \tag{2.3}$$

The left hand side of Eq. 2.3 is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\vec{r}}{dt} \cdot \frac{\partial f}{\partial \vec{r}} + \frac{d\vec{v}}{dt} \cdot \frac{\partial f}{\partial \vec{v}}$$

$$= \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_r f + \frac{\vec{K}}{m} \cdot \nabla_v f, \qquad (2.4)$$

where $\vec{K}$ is the external force, m is the mass of the individual gas particles, and

$$\frac{\partial}{\partial \vec{r}} \equiv \nabla_r = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}, \qquad (2.5)$$

and

$$\frac{\partial}{\partial \vec{v}} \equiv \nabla_v = \hat{i} \frac{\partial}{\partial v_x} + \hat{j} \frac{\partial}{\partial v_y} + \hat{k} \frac{\partial}{\partial v_z}, \qquad (2.6)$$

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit cartesian vectors.

The explicit form of the Boltzmann collision integral $(\frac{\partial f}{\partial t})_c$ is not of interest to us here, since the assumptions made in its derivation are not applicable to plasmas as discussed in Chapter 1. We will, therefore, proceed to a kinetic description of plasmas in the next section.

2.2 *The BBGKY Hierarchy of Equations*

Most of the kinetic theory of plasmas is based on one of the following three approaches :

(a) Fokker-Planck approach,

(b) BBGKY hierarchy of equations, and

(c) The method of diagrams.

In the Fokker-Planck approach, the statistical method for treating random processes is applied to ions or electrons. The fact that the probability of a large angle deflection of a charged particle in a plasma occuring as a succession of many small angle scatterings is much more than as a single binary scattering is used as the basis for the Fokker-Planck theory. The starting point in the other two approaches is the Liouville equation . In the method of diagrams, developed by Prigogine and Balescu [24,25], the Liouville equation is Fourier analysed and solved by iteration. In the BBGKY (Bogoliubov, Born, Green, Kirkwood and Yvon) method, a hierarchy of equations is derived from Liouville equation, which are then solved by successive approximations [2]. Since the work reported here is based on this method, we will discuss BBGKY system of equations in some detail.

2.2.1 *The Liouville Equation*

The microscopic state of a plasma or any other many-body system is completely defined by the coordinates and velocities of the particles. Similarly the statistical properties of such a system are completely determined by the spatial and velocity distribution of its particles. If a system has N-particles, the N-particle distribution function, F_N, is defined as follows:

$$F_N(\vec{q}, \vec{p}, t) \equiv F_N(\vec{q}_1, \vec{q}_2 \ldots, \vec{q}_N, \vec{p}_1, \vec{p}_2 \ldots, \vec{p}_N, t), \tag{2.7}$$

where $\vec{q}$ and $\vec{p}$ denote the coordinates and the momenta of the N-particles, and F_N is such that

$$V^{-N} F_N(\vec{q},\vec{p},t)\, d\vec{q}d\vec{p} \equiv V^{-N} F_N(\vec{q},\vec{p},t) d\vec{q}_1 d\vec{q}_2 \ldots d\vec{q}_N d\vec{p}_1 d\vec{p}_2 \ldots d\vec{p}_N) \tag{2.8}$$

is the probability that at time t the coordinates and velocities of the particles have the values $\vec{q}_1$, $\vec{q}_2 \ldots \vec{q}_N$, $\vec{p}_1$, $\vec{p}_2 \ldots \vec{p}_N$ in the range $d\vec{q}_1$, $d\vec{q}_2 \ldots d\vec{q}_N$, $d\vec{p}_1$, $d\vec{p}_2 \ldots d\vec{p}_N$, respectively. F_N is normalized according to the following relation :

$$\int \ldots \int F_N(\vec{q}, \vec{p}, t)\, d\vec{q}d\vec{p} = V^N. \tag{2.9}$$

The evolution of F_N in time and phase-space is governed by the Liouville equation [4,2]:

$$\frac{dF_N}{dt} = \frac{\partial F_N}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial F_N}{\partial \vec{q}_i} \cdot \dot{\vec{q}}_i + \frac{\partial F_N}{\partial \vec{p}_i} \cdot \dot{\vec{p}}_i\right) = 0 \tag{2.10}$$

where a dot on the top of the variable denotes derivative with respect to time.

By means of Canonical equations of motion [23],

$$\begin{aligned} \vec{v}_i \equiv \frac{\vec{p}_i}{m} = \dot{\vec{q}}_i \equiv \frac{\partial H_N}{\partial \vec{p}_i}, \\ \dot{\vec{p}}_i = -\frac{\partial H_N}{\partial \vec{q}_i}, \quad i = 1,2,\ldots,N, \end{aligned} \tag{2.11}$$

where H_N stands for the Hamiltonian of the N-particles,

$$H_N = \sum_{i=1}^{N} [T_i(\text{Kinetic energy}) + V_i(\text{Potential energy})], \tag{2.12}$$

We can write the Liouville equation in the form

$$\frac{\partial F_N}{\partial t} = [H_N, F_N], \tag{2.13}$$

or,

$$\frac{\partial F_N}{\partial t} + [F_N, H_N] = 0, \tag{2.14}$$

where

$$[H_N, F_N] = \sum_{i=1}^{N} \left(\frac{\partial H_N}{\partial \vec{q}_i} \cdot \frac{\partial F_N}{\partial \vec{p}_i} - \frac{\partial H_N}{\partial \vec{p}_i} \cdot \frac{\partial F_N}{\partial \vec{q}_i}\right) \tag{2.15}$$

is the Poisson-bracket expression for H_N and F_N. The dots between the derivatives on the r.h.s of Eq. (2.15) denote a scalar product of two vectors.

For an external force-free condition, which will be considered in this work,

$$H_N = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{1 \leq i < j \leq N} \phi_{ij}, \tag{2.16}$$

where $p_i = |\vec{p}_i|$, and ϕ_{ij} is the interparticle potential, a function of interparticle distance and not of the individual coordinates, that is,

$$\phi_{ij} = \phi_{ij}(|\vec{q}_i - \vec{q}_j|) \tag{2.17}$$

We define the following operators for an external force-free system of N-particles :

$$K_N = \sum_{i=1}^{N} \frac{\vec{p}_i}{m_i} \cdot \frac{\partial}{\partial \vec{q}_i} \equiv \sum_{i=1}^{N} \vec{v}_i \cdot v_{q_i}, \tag{2.18}$$

$$\Theta_N \equiv \sum_{1 \le i < j \le N} \left(\frac{\partial \phi_{ij}}{\partial \vec{q}_i} \cdot \frac{\partial}{\partial \vec{p}_i} + \frac{\partial \phi_{ij}}{\partial \vec{q}_j} \cdot \frac{\partial}{\partial \vec{p}_j} \right), \tag{2.19}$$

$$\equiv \frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{\partial \phi_{ij}}{\partial \vec{q}_i} \cdot \left(\frac{\partial}{\partial \vec{p}_i} - \frac{\partial}{\partial \vec{p}_j} \right). \tag{2.20}$$

In terms of these operators the Liouville equation can be written as

$$\frac{\partial F_N}{\partial t} + (K_N - \Theta_N) F_N = 0 \tag{2.21}$$

where $(K_N - \Theta_N)$ can be identified as the Hemiltonian operator for an external force free-system of N-particles [7].

2.2.2 The BBGKY Equations

For brevity of notation, we define $\vec{x}_i \equiv (\vec{q}_i, \vec{p}_i)$, $i = 1,2,\ldots,N$. In order to obtain the BBGKY equations from the Liouville equation, we define the reduced distribution functions, $F_s(\vec{x}_1 \ldots \vec{x}_s, t)$, $s = 1,2,\ldots,N$, by

$$F_s(\vec{x}_1,\vec{x}_2\ldots\vec{x}_s,t) = V^{-(N-s)} \int \ldots \int F_N(\vec{x}_1,\vec{x}_2\ldots\vec{x}_N,t)d\vec{x}_{s+1}..d\vec{x}_N \tag{2.22}$$

where V is the total volume of the system. In order to concentrate on volume effects and minimize boundary effects, the system is assumed to be infinite by allowing both N and V to become infinite in such a way that the particle density, $n = N/V$ = finite.

By integrating the Liouville equation over $\vec{x}_{s+1}\ldots\vec{x}_N$, it is straight forward to obtain the so-called BBGKY hierarchy of of N equations, which were derived independently by Bogoliubov, Born, Green, Kirkwood and Yvon [3,5,6].

$$\frac{\partial F_s}{\partial t} = [H_s, F_s] + \frac{N-s}{V} \int d\vec{x}_{s+1} \Big[\sum_{i=1}^{s} \phi_{i,s+1}, F_{s+1} \Big], \tag{2.23}$$

$$s = 1,2,\ldots,N,$$

where the square brackets denote the Poisson brackets (see Eq. 2.15). For a derivation of Eq. (2.23), reference should be made either to the original works cited above, or to Refs. [7 26].

Again considering external-force free system and by defining the operators :

$$K_s \equiv \sum_{i=1}^{s} \frac{\vec{p}_i}{m_i} \cdot \frac{\partial}{\partial \vec{q}_i}, \qquad (2.24)$$

$$\Theta_s \equiv \sum_{1 \le i < j \le s} \frac{\partial \phi_{ij}}{\partial \vec{q}_i} \cdot \left(\frac{\partial}{\partial \vec{p}_i} - \frac{\partial}{\partial \vec{p}_j}\right), \qquad (2.25)$$

$$\Theta_1 = 0, \qquad (2.25a)$$

and

$$L_s \equiv \sum_{i=1}^{s} \int d\vec{x}_j \frac{\partial \phi_{ij}}{\partial \vec{q}_i} \cdot \frac{\partial}{\partial \vec{p}_i}, \quad j=s+1, \qquad (2.26)$$

and taking the limit of $(N-s)/V = n$ as N and V tend to be infinite, we get the following alternative form of BBGKY equations :

$$\frac{\partial F_s}{\partial t} + (K_s - \Theta_s) F_s = n L_s F_{s+1}, \qquad (2.27)$$

$$s=1,2,..,N.$$

As pointed earlier $(K_s - \Theta_s)$ can be identified as the Hamiltonian operator for the external-force-free system of s particles and L_s is sometimes called the phase mixing operator [7].

2.3 *The Generalized Kinetic Equation, Irreversibility and Statistical Independence*

The equations (2.27) are equivalent to the Liouville equation, since they have been obtained from the latter without making any additional assumptions. However, it can be seen immediately that equations (2.27) form a chain : for every s, the equation for F_s contains the unknown F_{s+1}. The first of these equations, namely

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m_1}\cdot\frac{\partial}{\partial \vec{q}_1}\right)F_1(\vec{q}_1,\vec{p}_1,t) = n \int\int d\vec{q}_2 d\vec{p}_2 \frac{\partial \phi_{12}}{\partial \vec{q}_1}\cdot\frac{\partial F_2}{\partial \vec{p}_1} \tag{2.28}$$

can be called a generalized kinetic equation, applicable to any external-force-free system. The left number of this equation has exactly the same form as the left hand side of the Boltzmann equation. On the other hand, the right hand member of Eq. (2.28) gives an exact expression for the change of F_1 due to interactions. It cannot, however, be solved without making additional assumptions, because F_2 is unknown. If we write the equation for F_2, it will contain the unknown F_3, and so on.

Another very important feature of the BBGKY Eqs.(2.27) or the generalized kinetic Eq.(2.28) is that they are reversible in time, just as the Liouville equation from which they have been derived. Therefore, unless a definite direction of

time is introduced into these equations, they will not describe the irreversible approach to equilibrium from non-equilibrium states.

It may also be noted here that in terms of the reduced distribution functions, statistical independence of the particles can be defined as follows. If

$$F_s(\vec{x}_1,\ldots,\vec{x}_s,t) - \prod_{i=1}^{s} F_1(\vec{x}_i,\ t) = 0, \qquad (2.29)$$

$$s \geq 2,$$

the particles in the system are statistically independent or uncorrelated. This is also called the assumption of molecular Chaos [16]. This is one of the assumptions made in the derivation of Boltzmann equation. Since in a plasma, due to the long range nature of the Coulomb force, the particles are correlated i.e. Eq. (2.29) is not satisfied, a different approach is called for.

In Section 2.5, we present briefly, Bogoliubov's method for solving the BBGKY equations, our main interest being in the equilibrium one-particle and two-particle distribution functions for an external-force-free fully ionized plasma. In the first order approximation Bogoliubov's method solves the BBGKY equations for small values of the plasma parameter, ε, which is defined in Section 2.4.

2.4 *The Plasma Parameter and the Plasma Approximation*

The plasma parameter, ε, is a measure of the inverse of plasma particles in a Debye sphere and can be defined by [14] :

$$\varepsilon = \frac{1}{n\, r_D^3} \qquad (2.30)$$

where N is the particle density, and r_D is the Debye length given by [26, 27, 14, 2]

$$r_D = \left(\frac{\varepsilon_o kT}{ne^2 z} \right)^{1/2} = \left(\frac{\varepsilon_o \Theta}{ne^2 z} \right)^{1/2} \qquad (2.31)$$

In the last equation k is the Boltzmann constant, e is the absolute value of electron charge, T or Θ is the temperature defined in terms of the velocity distribution function in Chapter 3, and ε_o is the permittivity of free space (Rationalized M.K.S. or SI units are used in this work).

It can be easily seen that the plasma parameter, ε, is also a measure of the ratio of the mean interparticle potential energy to the mean kinetic energy of the particles in a plasma, that is

$$\varepsilon \sim \frac{<PE>}{<KE>} . \qquad (2.32)$$

If $\varepsilon \ll 1$, the interparticle potential energy will be much less than the mean kinetic energy of the particles. This will be so far a sufficiently dilute plasma. The assumption $\varepsilon \ll 1$ is called the plasma approximation. It may be noted that as the plasma particle density decreases, the number of particles in the Debye sphere increases, and then ε decreases.

2.5 *Bogoliubov's Approach to the Solution of BBGKY Equations*

The BBGKY hierarchy of equations can be solved by three different approaches :

1. Bogoliubov's Method,
2. Frieman and Sandri Method, and
3. Cluster Expansion.

As was mentioned in Chapter 1, the three approaches are equivalent. All the methods, in their first order approximations, solve the BBGKY equations for small values of the plasma parameter. We will discuss the Bogoliubov's method in some detail.

The assumptions made in Bogoliubov's theory are :

1. The evolution in time of all s-particle distribution functions, F_s, $s \geq 2$, can be described through their functional dependence on the one-particle distribution function $F_1(\vec{x}_1, t) \equiv F_1$:

$$F_s(\vec{x}_1, \ldots, \vec{x}_s, t) \rightarrow F_s(\vec{x}_1, \ldots, \vec{x}_s, F_1), \qquad (2.33)$$

$$s \geq 2,$$

With the help of this assumption the generalized kinetic equation can be written in the form

$$\frac{\partial F_1}{\partial t} = A(\vec{x}_1, F_1), \qquad (2.34)$$

where $A(\vec{x}_1 F_1)$ is a functional of F_1.

2. The past is defined as that direction of time in which correlations among the particles decrease to zero :

$$\lim_{t \to -\infty} [F_s(\vec{x}_1 \ldots \vec{x}_s, t) - \prod_{i=1}^{s} F_1(\vec{x}_i, t)] = 0, \qquad (2.35)$$

$$s \geq 2.$$

3. The following expansions in terms of the plasma parameter, ε, are made :

$$F_s = \sum_{n=0} \varepsilon^n F_s^{(n)}, \qquad s \geq 2. \qquad (2.36)$$

$$\frac{\partial F_1}{\partial t} = \sum_{n=0} \varepsilon^n A^{(n)} \qquad (2.37)$$

When we utilize Eqs. (2.33) to (2.37) in the BBGKY equations, we obtain equations to each order of ε, which can be solved in succession.

2.6 *The Equilibrium One-Particle and Two-Particle Distribution Functions*

Whether Bogoliubov's theory or other equivalent approaches describe adequately the irreversible approach of non-equilibrium plasmas (or other systems) to a uniquely defined equilibrium condition is not yet known. However, to the first order in ε, it is known to satisfy the Boltzmann H-theorem [7]. As an extension of this fact, it is generally assumed that the Bogoliubov's theory does describe an irreversible approach to equilibrium. Guernsey and Lenard[2] have obtained the following one-particle and two-particle equilibrium distribution functions, to the first order in ε,:

$$F_1(\vec{x}_1,t)_{\text{Equilibrium}} \equiv F^E(p) = (2\pi m\Theta)^{-3/2} \exp(-p^2/2m\Theta) \tag{2.38}$$

and

$$F_2(\vec{x}_1,\vec{x}_2,t)_{\text{Equilibrium}} \equiv F_2^E(\vec{r},\vec{p}_1,\vec{p}_2)$$

$$= F^E(p_1)F^E(p_2)[1 - \frac{z_1 z_2 e^2}{4\pi\varepsilon_o r\Theta} \exp(-\frac{r}{r_D})], \tag{2.39}$$

where

m = mass of an individual particle,

Θ = temperature at equilibrium,

$z_1 e$ = charge of particle 1,

$z_2 e$ = charge of particle 2, and

$r = |\vec{r}_1 - \vec{r}_2|$, the distance between particles 1 and 2

$= |\vec{q}_1 - \vec{q}_2|$.

The factor $[1 - \frac{z_1 z_2 e^2}{4\pi\varepsilon_o r\theta} \exp(-\frac{r}{r_D})]$ in Eq. (2.39) shows the effect of interparticle correlations in a plasma. We will discuss it further in Chapter 4. It should be noted that as required by basic definitions $F^E(p)$ in Eq. (2.38) normalizes to unity when integrated over $\vec{p}$ (see Appendix I).

We now proceed to Chapter 3, where we define the macroscopic variables, the flux vectors, the macroscopic conservation equations, the rate equations, and the transport coefficients. Then in Chapter 4, we discuss the hydrodynamic approximation for the distribution functions and obtain appropriate expressions for the transport coefficients considering small deviation from equilibrium.

Chapter - 3

MACROSCOPIC CONSERVATION EQUATIONS AND TRANSPORT COEFFICIENTS

3.1 The Macroscopic Variables

The macroscopic variables of interest in this work are the particle density $n(\vec{q},t)$ or the mass density $\rho(\vec{q},t)$, the macroscopic or mean velocity $\vec{u}(\vec{q},t)$ and the temperature $T(\vec{q},t)$ or $\theta(\vec{q},t)$. We wish to define these variables both in terms of the velocity distribution function $f(\vec{q}, \vec{v}, t)$ and the reduced one-particle distribution function $F_1(\vec{q}, \vec{p}, t)$ introduced in Chapter 2.

The following relationship between $f(\vec{q}, \vec{v}, t)$ and $F_1(\vec{q}, \vec{p}, t)$ holds from their respective definitions (Sections 2.1.1 and 2.2.2):

$$f(\vec{q},\vec{v},t)d\vec{v} = n F_1(\vec{q},\vec{p},t)d\vec{p} , \qquad (3.1)$$

where $n = N/V$, the particle density.

The macroscopic variables n, ρ, $\vec{u}$ and T are then straight forwardly defined in terms of $f(\vec{q}, \vec{v}, t)$ as

$$n(\vec{q},t) = \int f(\vec{q},\vec{v},t)\, d\vec{v} , \qquad (3.2)$$

$$\rho(\vec{q},t) = mn(\vec{q},t) , \qquad (3.3)$$

$$\vec{u}(\vec{q},t) = \frac{1}{n(\vec{q},t)} \int \vec{v}\, f(\vec{q},\vec{v},t)d\vec{v} \qquad (3.4)$$

and

$$\frac{3}{2} kT \equiv \frac{3}{2} \Theta = \frac{1}{n(\vec{q},t)} \int \frac{1}{2} mU^2 f(\vec{q},\vec{v},t) d\vec{v} \tag{3.5}$$

where $U \equiv |\vec{U}| = |\vec{v}-\vec{u}|$. $\vec{U}$ is called the thermal or peculiar velocity. Extension of Eqs. (3.1) to (3.5) to systems of more than one species is straight forward. It is easily seen that the equilibrium distribution function of Eq. (2.38) satisfies above definitions with $\vec{u} = 0$ (see Appendix I).

3.2 Flux Vectors

The flux vectors of interest here are the particle flux, $\vec{\Gamma}(\vec{q},t)$, the momentum flux (which is actually a tensor called pressure or stress tensor) $\overline{\overline{P}}(\vec{q},t)$, and the heat flux, $\vec{J}$, defined as follows :

$$\vec{\Gamma}(\vec{q},t) = \int \vec{v} f(\vec{q},\vec{v},t) d\vec{v}$$

$$= n(\vec{q},t) \vec{u}(\vec{q},t) \tag{3.6}$$

$$\overline{\overline{P}} = [P_{ij}] \quad i,j = 1,2,3,$$

$$P_{ij} = m \int U_i U_j f(\vec{q},\vec{v},t) d\vec{v} \tag{3.7}$$

$$\vec{J} = \frac{1}{2} m \int U^2 \vec{U} f(\vec{q},\vec{v},t) d\vec{v} \tag{3.8}$$

All the quantities appearing in Eqs. (3.6) to (3.8) have already been defined. It follows from the defination of $\overline{\overline{P}}$ and T that

$$P_{ii} = nkT = n\theta, \quad i = 1,2,3. \qquad (3.7a)$$

It should be pointed here that in a system with interparticle potential energy, the pressure tensor and the heat flux will, in general, have terms related to interparticle potentials. These terms have been derived by Choh and Uhlenbeck [2] and are very complicated to be quoted here. In the plasma approximation, $\varepsilon << 1$, these terms are negligible.

3.3 *The Macroscopic Conservation Equations*

The macroscopic conservation equations are obtained by multiplying the kinetic equation by the conserved quantities, namely mass m, momentum $m\vec{U}$ and the kinetic energy $\frac{1}{2}mv^2$, in turn, and integrating over $\vec{v}$. If the potential energy flow is negligible, these conservation equations take the following form for an external-force-free system [2,16]

$$\frac{\partial n}{\partial t} + \nabla\cdot(\vec{\Gamma}) = 0,$$

or,
$$\frac{\partial \rho}{\partial t} + \nabla\cdot(m\vec{\Gamma}) = 0, \qquad (3.9)$$

$$\rho\left[\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right]\vec{u} + \nabla\cdot\bar{\bar{P}} = 0, \qquad (3.10)$$

and
$$\left[\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right]\theta + \frac{2}{3n}\left[\nabla\cdot\vec{J} + \bar{\bar{P}}:\bar{\bar{D}}\right] = 0, \qquad (3.11)$$

where $\overline{\overline{D}}$ is the strain-rate-tensor, whose component are

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial q_i} + \frac{\partial u_j}{\partial q_i} \right). \qquad (3.11a)$$

The single dot product between a vector $\vec{A}$ and tensor $\overline{\overline{B}}$ is a vector whose components are

$$(\vec{A}\cdot\overline{\overline{B}})_i = \sum_j A_j B_{ji}.$$

The double dot product between two tensors $\overline{\overline{A}}$ and $\overline{\overline{B}}$ is a scalar given by

$$\overline{\overline{A}} : \overline{\overline{B}} = \sum_{i,j} A_{ij} B_{ji}.$$

The extension of Eqs.(3.9) to (3.11) to systems with more than one species is straight forward, provided the flux vectors defined in Section 3.2 are also rewritten for individual species.

3.4 *The Rate Equations and the Transport Coefficients*

The phenomenological rate equations for transfer of heat, momentum and mass (or number of particles) are :

1. Fourier's Equation of Heat Conduction :

$$\vec{J} = -K \nabla T \qquad (3.12)$$

2. Newton's Equation of Viscosity :

$$P_{ij} = -\mu \frac{\partial u_i}{\partial q_j}, \quad i \neq j. \tag{3.13}$$

3. Fick's Equation of Diffusion :

$$\vec{\Gamma} = -D \nabla n. \tag{3.14}$$

Equations (3.12) to (3.14) define the transport coefficients of interest in this work, namely, the coefficient of thermal conductivity, K, the coefficient of viscosity, μ, and the diffusivity or diffusion constant, D. It should be emphasized that we have assumed K, μ and D to be independent of direction. This will be the case if plasma is free from external fields.

In Chapter 4, the hydrodynamic approximation of the distribution functions and the generalized kinetic equation will be considered. It will be seen that if we retain only the first power of the gradients of the macroscopic variables, the moments of the kinetic equation yield the rate equations (3.12) to (3.14) together with the appropriate expressions for the coefficients K, μ and D, defined above. The expressions obtained are found to be in good agreement with the results of other authors based on more advanced theory, for example Fokker-Planck approach [21, 22] or a perturbation solution of BBGKY equations [2, 20].

Chapter - 4

TRANSPORT COEFFICIENTS IN THE HYDRODYNAMIC APPROXIMATION

4.1 The Hydrodynamic Approximation

In the hydrodynamic approximation [2,20] to the kinetic equation, it is assumed that the distribution functionsdo not depend on space and time explicitly. The spatial and time dependence is assumed to be only through a functional dependence of the distribution functions on macroscopic quantities $n(\vec{q},t)$, $\vec{u}(\vec{q},t)$ and $T(q,t)$. That is, we assume for F_1 ($\equiv F$) and F_2 the following forms:

$$F(\vec{x},t) \rightarrow F(\vec{p} \mid n, \vec{u}, T) \tag{4.1}$$

and

$$F_2(\vec{x}_1,\vec{x}_2,t) \rightarrow F_2(\vec{p}_1,\vec{p}_2,r_{12} \mid n_1,\vec{u}_1,T_1,n_2,\vec{u}_2,T_2), \tag{4.2}$$

where $r_{12} = |\vec{q}_1 - \vec{q}_2|$

It should be emphasized here that the functional dependence on n, $\vec{u}$ and T is understood to mean a dependence on n, $\vec{u}$ and T as well as on any of their spatial derivatives [7]. To state it explicitly, let $\vec{\beta}$ be a 5-component vector including n, $\vec{u}$ and T. The F and F_2 can be written as

$$F(\vec{x},t) \longrightarrow F(\vec{p}, \vec{\beta}, \nabla\vec{\beta}, \ldots\ldots) \tag{4.3}$$

and

$$F_2(\vec{x}_1,\vec{x}_2,t) \longrightarrow F_2(\vec{p}_1,\vec{p}_2,r_{12},\vec{\beta}_1,\vec{\beta}_2,\ \nabla\vec{\beta}_1,\ \nabla\vec{\beta}_2,\ldots), \tag{4.4}$$

where the dots indicate higher order spatial derivatives of $\vec{\beta}$. The iterative solution of Chapman-Enskog [1] and Choh-Uhlenbeck [2] for neutral gases and of Sundaresan-Wu [20] for fully ionized plasmas are based on the assumptions embodied in Eqs. (4.3) and (4.4).

4.1.1 Limitations of the Hydrodynamic Approximation

The hydrodynamic approximation is valid only if the system is not far from local thermodynamic equilibrium. Quantitatively, it means that the length over which fractional change in the macroscopic variables is significant is much larger than the mean free path, λ of the particles, that is

$$\lambda \frac{\nabla n}{n} << 1, \tag{4.5}$$

$$\lambda \frac{\partial u_i}{\partial q_j}\frac{1}{u_i} << 1, \tag{4.6}$$

and

$$\lambda \frac{\nabla T}{T} << 1 \tag{4.7}$$

Under these conditions, a particle will encounter approximately the same values of the macroscopic variables in its one free flight. The hydrodynamic approximation can then be mathematically

justified based on Hilbert's uniqueness theorem [7]. Thus, for a system not far from local thermodynamic equilibrium, the initial-value problem posed by the kinetic equation can be effectively replaced by an initial-value problem in hydrodynamic equations.

It should be pointed out here that retaining higher-order derivatives in Eqs. (4.3) and (4.4) or higher powers of first order derivatives, does not make the solutions obtained by this method applicable to systems with large gradients. It only refines the solutions further. More moments of the kinetic equation should be included if the system is far from equilibrium (see, for example Grad's method of thirteen moments [2]).

The above brief discussion provides some insight into the applicability of the concept of transport coefficients. Even if non-linear rate equations are used to define these coefficients, valid solutions can be obtained only for plasmas not far from local thermodynamic equilibrium. To illustrate it, we can cite the example of laser-pellet-fusion. The plasma created on the pellet-surface is far from local thermodynamic equilibrium. Thus in our view, no valid hydrodynamic model based on the five moments n, $\vec{u}$ and T can be developed. Either the kinetic equation should be solved as an initial value problem, or more moments should be included.

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4.2 *Characteristic Mean Free Path for a Plasma*

In a plasma, due to long range of Coulomb force, there is no well defined mean free path, because particles interact all the time. However, to define a state not far from local equilibrium through Eqs. (4.5) to (4.7), we can choose a suitable characteristic path length, for example, a path length over which a particle, by random walk, scatters through a total angle of 90° by multiple collisions. This path length for electrons in a fully ionized plasma is given by [14] :

$$\lambda_e \approx \left[\frac{nZ}{2\pi} \left(e^2/3\varepsilon_0 kT\right)^2 \ell n \Lambda \right]^{-1}, \qquad (4.8)$$

where Ze is the charge on ions and

$$\Lambda \equiv \frac{3}{2} \left(\frac{k^3 T^3}{\pi n}\right)^{1/2} \left(\frac{4\pi\varepsilon_0}{e^2 Z}\right)^{3/2}. \qquad (4.9)$$

$\ell n \Lambda$ is called the Coulomb logarithm.

Over a large range of kT and n, $\ell n \Lambda$ is found to be a weak function of plasma parameters [14]. The characteristic path length for ions, λ_i, is obtained upon replacing Z by Z^3 in Eq. (4.8). In writing Eq. (4.8), the Coulomb logarithm is assumed to be same for electron-electron, ion-ion and electron-ion collisions. This is a reasonable approximation due to slow variation of $\ell n \Lambda$. Λ given by Eq. (4.9) actually applies for electron-ion collisions. If this assumption is not made,

plasma ion and electron densities will appear separately in Eq. (4.8), and the corresponding expression for λ_e and λ_i can be easily written.

4.3 *Non-equilibrium Distribution Functions*

(a) One-Particle Distribution Function :

In accordance with the hydrodynamic approximation, if the plasma is not far from equilibrium, the spatial and temporal variations of non-equilibrium distribution functions are adequately described by their dependence on θ, $\vec{u}$ and n, and on their spatial gradients. For further development, we make the following assumptions :

1. Since the quantities $\lambda \frac{\nabla T}{T}$, $\frac{\lambda}{u_i} \frac{\partial u_i}{\partial q_j}$ and $\lambda \frac{\nabla n}{n}$ are small ($<< 1$), their higher powers can be neglected.
2. Without loss of generality, the direction of a gradient is taken to coincide with the x-direction.
3. Again, without loss of generality, the gradients in macroscopic variables are treated one by one to study the particular transport process associated with that gradient. For example, to study thermal conductivity, a temperature gradient is assumed to exist, while gradients in the macroscopic velocity or density are taken to be zero.

With the help of these assumptions, the one-particle distribution function for a non-equilibrium situation can be written as

$$F(\vec{p},\beta,\frac{\partial\beta}{\partial x}) = F^{E}(p)[1 - \frac{\lambda}{\beta}\cdot\frac{\partial\beta}{\partial x}\, h_{\beta}(\vec{p})], \qquad (4.10)$$

where $F^{E}(p)$ is the Maxwellian distribution given by Eq.(2.38), β is to be identified with any component of $\vec{\beta}$, in particular, θ, u_y and n respectively, and $h_{\beta}(\vec{p})$ is to be obtained by substituting Eq. (4.10) in the kinetic equation. The negative sign of the last term in Eq. (4.10) has been choosen for convenience. The function $h_{\beta}(\vec{p})$ must satisfy the following properties :

1. It must be a dimensionless function of $\vec{p}$, i.e.

$$h_{\beta}(\vec{p}) = h_{\beta}(\vec{p}') \qquad (4.11)$$

$$\text{where } \vec{p}' \equiv \vec{p}/\sqrt{m\theta} \qquad (4.12)$$

2. It follows from basic definitions that

$$\int_{-\infty}^{\infty} F\cdot d\vec{p} = 1$$

Since

$$\int_{-\infty}^{\infty} F^{E}\, d\vec{p} = 1$$

(see Appendix I), we must have,

$$\int_{-\infty}^{\infty} h_{\beta}(\vec{p})\, d\vec{p} = 0. \qquad (4.13)$$

Obviously, Eq. (4.13) is also satisfied if we replace $\vec{p}$ by $\vec{p}'$.

(b) Two-Particle Distribution Function:

It can be easily seen that the generalized kinetic equation applicable to plasma (Eq. 2.28) contains not only F_1, but also the two-particle distribution function F_2. Neither the solution nor the moments of the kinetic equation can be obtained without knowledge of F_2. By analogy with the equilibrium two-particle distribution function (Eq. 2.39), we assume that for a situation not far from equilibrium, F_2 can be written as

$$F_2(\vec{p}_1,\vec{p}_2,r,\vec{\beta}_1,\vec{\beta}_2,\ldots) = F(\vec{p}_1,\vec{\beta}_1,\ldots)F(\vec{p}_2,\vec{\beta}_2,\ldots)G(r,\bar{\theta},\bar{n}) \tag{4.14}$$

where F is given by Eq.(4.10) and

$$G(r,\bar{\theta},\bar{n}) = 1 - \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_o r\bar{\theta}} \exp(-\frac{r}{\bar{r}_D}), \tag{4.15}$$

where,

$$r = r_{12}$$

$$\bar{\theta} = (\theta_1+\theta_2)/2$$

$$\bar{n} = (n_1+n_2)/2.$$

$\bar{r}_D$ is to be obtained by replacing θ and n in Eq. (2.31) by $\bar{\theta}$ and $\bar{n}$.

It may be pointed out here that F_2 has a singularity at $r = 0$. This can not be removed unless quantum effects are accounted for. For a detailed discussion reference should be made to [2]. It should also be emphasized that the equilibrium form of F_2 (Eq. 2.39) is applicable only for dilute plasma ($\varepsilon << 1$). Hence the results obtained by taking small deviations around this form will also be applicable only to dilute plasma.

Even with the above simplifying assumptions, it can be easily seen that the integral on R.H.S. of Eq. (2.28) can not be performed analytically. If $\nabla\theta(\equiv k\nabla T)$ and ∇n are small, the function $G(r,\bar{\theta},\bar{n})$ can be expanded in a Taylor series around θ_1 and n_1. Again only the first order terms in the gradients are retained. The detailed expansion is given in Appendix II.

4.4 *Collision Term in the Generalized Kinetic Equation*

With the approximations made about the form of F_2 for a non-equilibrium situation in Section 4.3b and Appendix II, the collision integral in the generalized kinetic equation, Eq.(2.28), can be easily evaluated. Considering only a fully ionized two species plasma, we have

$$\phi_{12} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_o r}, \qquad (4.16)$$

where $Z_1 e$ is the charge on particle 1 and $Z_2 e$ is the charge on particle 2. It should be noted that the particles 1 and 2

could belong either to the same species or different species. Thus there will be two terms in the collision integral, obtained by fixing Z_1 and varying Z_2 over the two species present. This summation can be performed at the end, after the collision term has been evaluated.

Denoting the collision integral in Eq. (2.28) by I and utilizing Eqs. (4.14) to (4.16), we get,

$$I = n \frac{\partial F(1)}{\partial \vec{p}_1} \cdot \int_{-\infty}^{\infty} d\vec{q}_2 \frac{\partial \phi_{12}}{\partial \vec{q}_1} G(r,\bar{\theta},\bar{n}) \int_{-\infty}^{\infty} d\vec{p}_2 F(2), \tag{4.17}$$

where F(1) and F(2) denote F at points 1 and 2 respectively. By definition, the last integral in Eq. (4.17) is unity. The other integral in Eq. (4.17) is evaluated in Appendix III. The result is

$$I = \left[\frac{\pi a^2 r_D Z_1^2 Z_2^2}{3\theta} \left(\frac{\nabla\theta}{\theta} + \frac{\nabla n}{n} \right) \right] \cdot \frac{\partial F}{\partial \vec{p}} n \tag{4.18}$$

where $a = e^2/4\pi\varepsilon_o$.

The subscript 1 has been dropped, since it is no longer necessary. The generalized kinetic equation for electrons now takes the following form :

$$\frac{\partial F}{\partial t} + \frac{\vec{p}}{m_e} \cdot \frac{\partial F}{\partial \vec{q}} = \frac{\pi a^2 r_D}{3\theta} (n_e + n_i Z^2) \frac{\partial F}{\partial \vec{p}} \cdot \left[\frac{\nabla\theta}{\theta} + \frac{\nabla n}{n} \right]$$

$$= \frac{\pi a^2 r_D n Z}{3\theta} \frac{\partial F}{\partial \vec{p}} \cdot \left[\frac{\nabla\theta}{\theta} + \frac{\nabla n}{n} \right], \tag{4.19}$$

where m_e is the electron-mass, n_e and n_i are the average electron and ion densities, respectivoly, and all other quantities have already been defined. In the last step above, we have used the equality

$$n_e + n_i Z^2 = n Z \qquad (4.20)$$

which is readily identified.

It can be easily seen that the kinetic equation for ions can be obtained by replacing m_e by m_i and Z by Z^3 in Eq. (4.19).

By substituting F from Eq. (4.10) into Eq. (4.19), the governing equation for h_β can be obtained. It should be noted that with F from Eq. (4.10)

$$\frac{\partial F}{\partial t} \simeq \frac{\partial F}{\partial \beta} \cdot \frac{\partial \beta}{\partial t}, \qquad (4.21)$$

and

$$\frac{\partial F}{\partial \vec{q}} = \frac{\partial F}{\partial \beta} \cdot \nabla \beta, \qquad (4.22)$$

Since derivatives of β with respect to time or space can be neglected as higher-order terms. The resulting equations for h_β are then to be solved together with the macroscopic conservation equations and other defining equations for macroscopic variables and flux vectors given in Chapter 3. It is seen in the next section that h_β appears in the transport coefficients only through a definite integral. Since h_β is a

dimensionless function, this integral effects the transport coefficients only by a multiplicative constant, which will not be evaluated in this work.

4.5 *Derivation of Expressions of Transport Coefficients*

The expressions for the transport coefficients can now be obtained by taking moments of Eq. (4.19). This means multiplying Eq. (4.19) by the particle density (n_e or n_i), momentum ($n_e m_e \vec{U}$ or $n_i m_i \vec{U}$) and the kinetic energy ($\frac{1}{2} n_e m_e U^2$ or $\frac{1}{2} n_i m_i U^2$), in turn, and integrating over all $\vec{p}$. The following identities[2] are utilized in obtaining the moments :

$$\int Q(\vec{p}) \frac{\vec{p}}{m} \cdot \frac{\partial F}{\partial q} \, d\vec{p} = \nabla \cdot \int Q(\vec{p}) \frac{\vec{p}}{m} F \, d\vec{p}, \qquad (4.23)$$

and

$$\int Q(\vec{p}) \frac{\partial F}{\partial p} \, d\vec{p} = - \int F \frac{\partial Q}{\partial \vec{p}} \, d\vec{p}, \qquad (4.24)$$

where $Q(\vec{p})$ is any property associated with $\vec{p}$. In writing Eq. (4.24) it is assumed that $FQ \rightarrow 0$ as $\vec{p} \rightarrow \pm \infty$.

4.5.1 *Ion Diffusivity*

Even though electrons are more mobile than ions, it is the diffusion coefficient of ions that governs net diffusion in plasma. This is due to ambipolar effects [27]. To investigate the ambipolar effects, the kinetic equation has to be solved in conjunction with Maxwell's equations. It can be proved that if $T_e = T_i$ and $m_i >> m_e$, ambipolar diffusion coefficient in twice the ion diffusivity [27].

To obtain an expression for ion diffusivity, we write Eq. (4.19) for ions with

$$\nabla\theta = 0, \qquad \nabla\vec{u} = 0, \qquad \nabla n \neq 0.$$

Under these conditions, the kinetic equation is

$$\frac{\partial F}{\partial t} + \frac{\vec{p}}{m_i} \cdot \frac{\partial F}{\partial \vec{q}} = \frac{\partial F}{\partial n}\frac{\partial n}{\partial t} + \left(\frac{\vec{p}}{m_i} \cdot \nabla n\right)\frac{\partial F}{\partial n}$$

$$= \frac{\pi a^2 r_D n Z^3}{3\theta} \frac{\partial F}{\partial \vec{p}} \cdot \frac{\nabla n}{n} . \qquad (4.25)$$

Multiplying Eq. (4.25) by $Q = n_i$, and utilizing Eqs. (4.23) and (4.24), we get Eq. (3.9) for the conservation of particles. The x-component of the particle current $\vec{\Gamma}$ is given by Eqs. (3.6), (3.1) and (4.10) as

$$\Gamma_x = \int \frac{p_x}{m_i} n_i F(\vec{p}, n, \nabla n) d\vec{p}$$

$$= \int \frac{p_x}{m_i} n_i F^E(\vec{p})\left[1 - \frac{\lambda_i}{n_i}\frac{\partial n_i}{\partial x} h_n(\vec{p})\right] d\vec{p}$$

$$= -\frac{\lambda_i}{m_i}\frac{\partial n_i}{\partial x}(2\pi m_i\theta)^{-3/2} \int_{\infty}^{\infty} p_x e^{-p^2/2m_i\theta} h_n(\vec{p}) d\vec{p}$$

(4.26)

since $\int_{-\infty}^{\infty} p_x F^E(\vec{p}) d\vec{p} = 0$ (see Appendix I).

By changing the variable of integration from $\vec{p}$ to $\vec{p}' = \vec{p}/\sqrt{m\theta}$ and noting

$$d\vec{p} = (m_i\theta)^{3/2} d\vec{p}' , \qquad (4.27)$$

and using λ_i from Section (4.2), we get

$$\Gamma_x = - \frac{9\varepsilon_o^2}{\sqrt{2\pi}\,(Z+1)Z^3 \ell n \Lambda} \cdot \frac{\theta^{5/2}}{\sqrt{m_i}\, n_i\, e^4} I_n, \qquad (4.28)$$

where we have used the fact that $n = n_i(Z+1)$, and I_m is given by

$$I_n = \int_{-\infty}^{\infty} p_x' \, e^{-(p')^2/2} \, h_n(\vec{p}')\, d\vec{p}'. \qquad (4.29)$$

Comparing Eq. (4.28) with the rate Eq. (3.14) we get

$$D_i = \frac{9\epsilon_o^2 I_n}{\sqrt{2\pi}(Z+1)\, Z^3 \ell n \Lambda} \left(\frac{\theta^{5/2}}{\sqrt{m_i}\, n_i\, e^4} \right) \qquad (4.30)$$

Since I_n is a dimensionless quantity, the functional dependence of D_i on plasma parameters is given by the last term in Eq. (4.30).

It can be seen from Eq. (4.30) that the diffusion coefficient is inversely proportional to the square root of mass. Hence the electron diffusivity would be much larger than ion diffusivity. However, because of ambipolar effects, the much slower ion diffusion governs the net diffusion rate in the plasma.

4.5.2 Ion Viscosity

At or near equilibrium in a fully ionized plasma, the ions and electrons carry approximately the same amount of energy per particle. However, since the ions are much more massive than electrons, most of the momentum is carried by ions. Hence viscous effects are mainly due to ions. In this section, we will develop an expression for ion viscosity.

For this purpose we assume that the density and temperature are uniform throughout the plasma, and that the deviation from equilibrium is due to small gradients in the macroscopic velocity, i.e.,

$$\nabla n = 0 = \nabla\theta, \qquad \nabla \vec{u} \neq 0.$$

Further, to choose a one-dimensional situation, we assume $\nabla u_y \neq 0$, $\nabla u_x = 0 = \nabla u_z$. Under these conditions, the kinetic equation, Eq. (4.19), along with Equations (4.21) and (4.22), becomes

$$\frac{\partial F}{\partial t} + \frac{\vec{p}}{m_i} \cdot \frac{\partial F}{\partial \vec{q}} = \frac{\partial F}{\partial u_y} \frac{\partial u_y}{\partial t} + \frac{p_x}{m_i} \frac{\partial F}{\partial u_y} \frac{\partial u_y}{\partial x} = 0 \quad (4.31)$$

It is straight forward to see that by multiplying Eq. (4.31) by $Q = n_i m_i \vec{v}$, and integrating over all $\vec{p}$, we obtain Eq. (3.10) for the conservation of momentum. The components of the stress tensor are given by Eq. (3.7) as :

$$P_{xy} = \frac{1}{m_i} \int_{\infty}^{\infty} \not{p}_x \not{p}_y \, n_i \, F \, d\vec{\not{p}}, \tag{4.32}$$

where

$$\vec{\not{p}} = m_i \vec{U} = m_i (\vec{v} - \vec{u}). \tag{4.33}$$

Utilizing the assumed form of F for near-equilibrium conditions from Eq. (4.10), we obtain,

$$P_{xy} = \frac{n_i}{m_i} \int \not{p}_x \not{p}_y \, F^E \left[1 - \frac{\lambda_i}{u_y} \frac{\partial u_y}{\partial x} h_u(\vec{p})\right] d\vec{p} \tag{4.34}$$

It should be noted here that in the presence of macroscopic velocity, F^E is to be taken as :

$$F^E = (2\pi m\theta)^{-3/2} \exp(-\not{p}^2/2m\theta). \tag{4.35}$$

It is then easily seen that

$$\int_{-\infty}^{\infty} \not{p}_x \not{p}_y \, F^E \, d\vec{p} = 0 \quad \text{(see Appendix I)}$$

Thus we have,

$$P_{xy} = - \frac{n_i \lambda_i}{m_i} \frac{1}{u_y} \frac{\partial u_y}{\partial x} (2\pi m_i \theta)^{-3/2} \int_{-\infty}^{\infty} \not{p}_x \not{p}_y \, e^{-\not{p}^2/2m_i\theta} \, h_u(\vec{\not{p}}) \, d\vec{\not{p}} \tag{4.36}$$

By defining the nondimensional quantities,

$$\vec{p}' = \frac{\vec{p}}{\sqrt{m_i \theta}}, \quad u_y' = \frac{u_y}{\sqrt{\theta/m_i}},$$

and noting that

$$d\vec{p} = (m_i\theta)^{3/2} \, d\vec{p}',$$

We obtain,

$$P_{xy} = - \frac{n_i \lambda_i \sqrt{m_i \theta} \, I_u}{(2\pi)^{3/2}} \frac{\partial u_y}{\partial x} \tag{4.36a}$$

where I_u is the dimensionless integral given by

$$I_u = \frac{1}{u_y'} \int_{-\infty}^{\infty} p_x' \, p_y' \, e^{-(p')^2/2} \, h_u(\vec{p}) \, d\vec{p} \tag{4.37}$$

Using λ_i from Section (4.2), and noting that $n = n_i(Z+1)$, we get

$$P_{xy} = - \frac{9\varepsilon_o^2}{\sqrt{2\pi}} \frac{I_u}{Z^3(Z+1) \ln\Lambda} \frac{\sqrt{m_i} \, \theta^{5/2}}{e^4} \frac{\partial u_y}{\partial x} . \tag{4.38}$$

Comparing Eq. (4.38) with the rate Eq. (3.13), we obtain the desired expression for the coefficient of ion viscosity:

$$\mu = \frac{9\varepsilon_o^2}{\sqrt{2\pi}} \frac{I_u}{Z^3(Z+1)\ln\Lambda} \frac{\sqrt{m_i} \, \theta^{5/2}}{e^4} \tag{4.39}$$

It can be easily seen that the viscosity is directly proportional to the square root of mass. Hence viscous effects due to electrons will be negligible. The expression derived in Eq. (4.39) is in good agreement with that given by Fokker-Planck theory [9, 21].

4.5.3 Thermal Conductivity

The phenomenon of thermal conduction in a fully ionized plasma arises mainly due to collisions of electrons with ions or other electrons. Hence we will develop in this section an expression for electron thermal conductivity. The method is very similar to that used in Section (4.5.1) and (4.5.2) for diffusivity and viscosity. Here, we assume that the departure fr from equilibrium is due to a small gradient in temperature alone, i.e.,

$$\nabla\theta \neq 0, \quad \nabla n = 0 = \nabla\vec{u} = \vec{u}.$$

It may be noted that by setting $\vec{u} = 0$, we make $\vec{p} = \vec{p}$ for convenience. Under these conditions, the kinetic equation for electrons can be written as, from Eq. (4.19):

$$\frac{\partial F}{\partial t} + \frac{\vec{p}}{m_e}\cdot\frac{\partial F}{\partial \vec{q}} = \frac{\partial F}{\partial \theta}\frac{\partial \theta}{t} + \left(\frac{\vec{p}}{m_e}\cdot\nabla\theta\right)\frac{\partial F}{\partial \theta}$$

$$= \frac{\pi a^2 r_D n Z}{3\theta}\frac{\partial F}{\partial \vec{p}}\cdot\frac{\nabla\theta}{\theta} \qquad (4.40)$$

It is easily seen that if we multiply Eq. (4.40) by $Q = \frac{n_e p^2}{2m_e}$, and integrate over $d\vec{p}$, we obtain Eq. (3.11) for the conservation of energy. The integral on the right hand side of Eq. (4.40) will drop to zero by virtue of Eq. (4.24). The x-component of the heat flux vector $\vec{J}$ is given by Eq. (3.8) as :

$$J_x = \frac{n_e}{2m_e^2} \int_{-\infty}^{\infty} p^2 \vec{p} F \, d\vec{p} \qquad (4.41)$$

Utilizing F from Eq. (4.10), we obtain,

$$J_x = - \frac{\lambda_e n_e}{2 m_e^2 \Theta} \frac{\partial \Theta}{\partial x} \int_{-\infty}^{\infty} p^2 \vec{p} F^E h_\Theta (\vec{p}) \, d\vec{p} \qquad (4.42)$$

By using the Maxwellian distribution for F^E, and substituting $\vec{p}' = \frac{\vec{p}}{\sqrt{m\Theta}}$ as in Sections (4.5.1) and (4.5.2), we get :

$$J_x = - \frac{n_e \lambda_e \sqrt{m_e \Theta} \; I_\Theta}{(2\pi)^{3/2} (2 m_e)} \frac{\partial \Theta}{\partial x} , \qquad (4.43)$$

where

$$I_\Theta = \int_{-\infty}^{\infty} (p')^2 \vec{p}' e^{-(p')^2/2} h_\Theta (\vec{p}') \, d\vec{p}' \qquad (4.44)$$

Using λ_e from Eq. (4.8) and comparing Eq. (4.44) with the rate Eq. (3.12), we obtain the desired expression for thermal conductivity, K:

$$K = \frac{9 \varepsilon_o^2}{2\sqrt{2\pi}} \frac{I_\Theta}{(Z+1) \ln \Lambda} \cdot \frac{k \Theta^{5/2}}{m_e e^4} \qquad (4.45)$$

This is found to be in agreement with the results of Spitzer and Harm[21,22] based on Fokker-Planck theory and also in agreement with the dominant term in the expression obtained by Sundaresan

and Wu [20] by a first order perturbation solution of BBGKY equations.

The next chapter summarizes the present work reiterating the results and conclusions.

Chapter - 5

SUMMARY, CONCLUSIONS and RECOMMENDATIONS

5.1 Summary and Conclusions

In this work, transport coefficients of external field-free fully ionized plasmas have been investigated. The transport coefficients studied are diffusion constant, viscosity and thermal conductivity. The study is based on a kinetic description of plasma. The results obtained are found to be in good agreement with those obtained by other authors based on different approaches, mainly Fokker-Planck approach.

In Chapter 1, the various models available for the analysis of plasma behaviour were reviewed, and a summary of background literature was presented. The phenomenological fluid approach and the kinetic approach were compared in some detail. The inadequacy of the Boltzmann kinetic equation for plasmas was pointed out. This is mainly due to long-range nature of the Coulomb force encountered in plasmas.

The alternative kinetic approach, based on BBGKY hierarchy of equations derived from Lioville equation, is then presented in detail in Chapter 2. The problems of multiple collisions, and inter-particle correlations (statistical dependence), both arising mainly due to long range of the Coulomb force, were discussed. Finally Bogoliubov's approach to the solution of BBGKY equations was discussed.

In Chapter 3, appropriate moments of the kinetic equation are obtained leading to the definition of macroscopic variables, flux vectors and the conservation equations. The rate equations are introduced and transport coefficients defined.

Chapter 4 presents our work. In this chapter, a simple method to obtain expressions for transport coefficients is developed. The method starts with the recognition of following facts:

(1) The concept of transport coefficients is valid for near-equilibrium situations, i.e., for systems in which deviation from equilibrium is small. Furthermore, for the concept to be of any practical use, the transport coefficients must not depend on the details of the deviation. Also due to the smallness of the deviation, only the first power of the magnitude of deviation need be considered.

(2) Since the details of the deviation from equilibrium are not important, the final expressions for the transport coefficients are essentially a result of equilibrium distribution functions.

This is brought out in the present work. We start with small deviations from one-particle and two-particle distribution functions [Eqs. 4.10 and 4.14]. These when substituted into the first of BBGKY hierarchy of equations [Eq. 2.23], lead to the generalized kinetic equation [Eq. 4.19] which is no longer coupled with the remaining equations in the hierarchy. It is then possible to take the moments of the kinetic equation, once

again leaving out the details of the nature of deviation from equilibrium, represented by the functions $h_\beta(\vec{p})$ in Eq. 4.10. Comparison with the rate equation then immediately leads to appropriate expressions for the transport coefficients. These coefficients as obtained in this work [Eqs. 4.30, 4.39, and 4.45] can be summarized as follows:

1. Diffusion constant, D

$$D \propto \frac{\theta^{5/2}}{\sqrt{m_i}\, n_i\, e^4} \qquad (5.1)$$

2. Viscosity, μ

$$\mu \propto \frac{\sqrt{m_i}\, \theta^{5/2}}{e^4} \qquad (5.2)$$

3. Thermal Conductivity, K

$$K \propto \frac{k\, \theta^{5/2}}{\sqrt{m_e}\, e^4} \qquad (5.3)$$

These expressions are in agreement with the work of other authors based on Fokker-Planck equation [21, 22, 9]. In case of thermal conductivity, Sundaresan and Wu [20] have derived an expression following Bogoliubov's approach to the solution of BBGKY equations. The expression obtained in this work agrees with the dominant term in their expression.

The constants of proportionality in Eqs. (5.1) to (5.3) can, in principle, be determined by solving the kinetic equation for $h_\beta(\vec{p})$ and then determining the integrals I_n, I_u and I_θ in Eqs. (4.30), (4.39) and (4.45), respectively. This, however, is very lengthy and mathematically involved task, as is indicated by the work of Sundaresan and Wu [20]. It should be noted that once the expressions in Eqs.(5.1) to (5.3) have been obtained, the constant of proportionality can be determined by a single experimental measurement in each case. Reference to Eqs. (4.30), (4.39) and (4.45) indicates that the constants of proportionality will, in general, be functions of Z. This should be kept in mind if the same are to be obtained experimentally.

5.2 *Suggestions for Future Work*

Instead of suggesting minor variations of what has been reported here, we will end this work by pointing out areas in which further progress will be of maximum value :

1. For plasmas with large gradients in the macroscopic quantities, for example, the plasma on the surface of a laser irradiated fusion-pellet, the hydrodynamic approach based on the first five moments of the distribution function, namely n, $\bar{u}$ and T, is not valid. Hence a method based on either additional moments or a direct solution of the initial value kinetic problem should be attempted. It should be noted that the existing hydrodynamic codes for laser-pellet-fusion circumvent

this difficulty by empirical concepts like heat-inhibition factors etc. This appears to be unsatisfactory.

2. The classical description of plasma does not seem to be satisfactory due to divergence problem at small inter-particle distances. Quantum effects should be included.

3. Methods should be extended to partially ionized plasmas, with many species. This will necessitate the modelling of ionization-recombination behavior and charged-neutral as well as neutral-neutral collisions, which can be a very complicated task to perform satisfactorily.

4. External fields, either in the form of steady electric and magnetic fields or electromagnetic radiation, or something in-between, should be considered.

5. Relativistic effects in the plasma, necessitating the consideration of interparticle magnetic interaction, the plasma radiation, and its interaction with plasma particles, should be included.

6. It is impossible to keep a plasma free from collective phenomena(linear and nonlinear waves). In fact the collective mechanisms can be dominant modes for transfer of mass, momentum and energy, and can manifest in the form instabilities. The list of collective modes that can be excited in a plasma is very large. Methods to specify the collective modes that are likely to be excited under given conditions and their effect on transport phenomena in a plasma should be included in any realistic analysis.

7. The problems related to turbulence and fluctuations in plasmas have been attacked only recently and there is sufficient scope for further work.

8. There are problems of more basic nature, which can be attacked. These relate to the construction of an irreversible statistical or kinetic theory based on a reversible dynamic theory, insufficiency of the statistical element in the distribution functions, etc. These problems need to be investigated further.

9. Finally, any theory for dense plasmas is likely to be very complicated. It will, therefore, be necessary to supplement on substantiate theoretical effort by experimental work aimed at measuring plasma properties under various conditions.

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Appendix I

Properties of Equilibrium Distribution Function

1. One-Particle Equilibrium Distribution Function

$$F^E = (2\pi m\theta)^{-3/2} \exp(-\not{p}^2/2m\theta),$$

where

$$\vec{\not{p}} \equiv \vec{p} - m\vec{u}, \quad \not{p} = |\vec{\not{p}}|.$$

The following properties of F^E are easily verified:

$$\int_{-\infty}^{\infty} F^E \, d\vec{p} = 1, \tag{I-1}$$

$$\int_{-\infty}^{\infty} \vec{\not{p}} F^E \, d\vec{p} = 0, \tag{I-2}$$

$$\int_{-\infty}^{\infty} \frac{\not{p}^2}{2m} F^E \, d\vec{p} = \frac{3}{2}\theta, \tag{I-3}$$

$$\int_{-\infty}^{\infty} \not{p}_x \not{p}_y F^E \, d\vec{p} = 0, \tag{I-4}$$

$$\int_{-\infty}^{\infty} \not{p}^2 \not{p}_x F^E \, d\vec{p} = 0, \text{ etc.} \tag{I-5}$$

In th : above integrals, it may be noted that

$$d\vec{p} = dp_x \, dp_y \, dp_z = d\vec{\not{p}} = d\not{p}_x \, d\not{p}_y \, d\not{p}_z$$

The properties (I-1) to (I-5) can be verified either in the Cartesian coordinates or by switching over to spherical polar coordinates.

By definition, we must have

$$\int F_2 \, d\vec{p}_2 \, d\vec{q}_2 = V F(1)$$

It is easily seen that the equilibrium two-particle distribution function given by Eq. (2.39) satisfies the above identify if $N >> 1$, which will always be the case.

Appendix II

Taylor Series Expansion of $G(r, \bar{\theta}, \bar{n})$

Let $\frac{e^2}{4\pi\varepsilon_o} \equiv a$, then,

$$G(r, \theta, n) = 1 - \frac{aZ_1Z_2}{\theta r} \exp(-\frac{r}{r_D}), \qquad \text{(II.1)}$$

where

$$r_D = \left(\frac{\varepsilon_o \theta}{ne^2}\right)^{1/2} \quad \text{(see Eq. 2.31).}$$

By direct partial differentiation, we obtain,

$$\frac{\partial G}{\partial \theta} = \frac{aZ_1Z_2}{\theta^2 r}\left[1 - \frac{r}{2r_D}\right] \exp\left(-\frac{r}{r_D}\right), \qquad \text{(II.2)}$$

and

$$\frac{\partial G}{\partial n} = \frac{aZ_1Z_2}{2\theta r_D n} \exp\left(-\frac{r}{r_D}\right). \qquad \text{(II .3)}$$

Now, we write,

$$G(r, \bar{\theta}, \bar{n}) \simeq G(r, \theta_1, n_1) + \left.\frac{\partial G}{\partial \theta}\right|_{\theta_1, n_1} (\bar{\theta} - \theta_1) + \left.\frac{\partial G}{\partial n}\right|_{\theta_1, n_1} (\bar{n} - n_1), \qquad \text{(II .4)}$$

where

$$\bar{\theta} - \theta_1 = \frac{\theta_2 - \theta_1}{2} \simeq \frac{1}{2} \nabla\theta \cdot \vec{r} = \frac{1}{2}\frac{\partial \theta}{\partial x}\, x, \qquad \text{(II .5)}$$

and

$$\bar{n}-n_1 = \frac{n_2-n_1}{2} \simeq \frac{1}{2} \nabla n \cdot \vec{r} = \frac{1}{2} \frac{\partial n}{\partial x} x, \qquad (II.6)$$

where x denotes the x-component of $\bar{r}$.

Substituting Eqs. (II.2), (II.3), (II.5) and (II 6) into Eq. (II.4), we get the desired expansion for $G(r, \bar{\theta}, \bar{n})$ as follows:

$$G(r, \bar{\theta}, \bar{n}) = 1 - \frac{aZ_1Z_2}{r\theta_1} \exp\left(-\frac{r}{r_{D1}}\right)$$

$$+ \frac{aZ_1Z_2}{2\theta_1^2} \left[1 - \frac{r}{2r_{D1}}\right] \exp\left(-\frac{r}{r_{D1}}\right) \frac{\partial \theta}{\partial x}\Big|_1 \frac{x}{r}$$

$$+ \frac{aZ_1Z_2}{4\theta_1 n_1} \exp\left(-\frac{r}{r_{D1}}\right) \frac{\partial n}{\partial x}\Big|_1 \frac{x}{r_{D1}}, \qquad (II.7)$$

where,

$$r_{D1} = \left(\frac{\varepsilon_o \theta_1}{n_1 e^2 Z}\right)^{1/2}.$$

Thus $G(r, \bar{\theta}, \bar{n})$ has been expressed in terms of the macroscopic variables at point 1 and their derivatives at the same point.

Appendix III

Evaluation of the Collision Integral in Eq. (4.17)

In this appendix we will show that Eq. (4.17), namely

$$I = n \frac{\partial F(1)}{\partial \vec{p}_1} \int_{-\infty}^{\infty} d\vec{q}_2 \frac{\partial \phi_{12}}{\partial \bar{q}_1} G(r, \bar{\theta}, \bar{n}) \int_{-\infty}^{\infty} d\vec{p}_2 F(2), \tag{III.1}$$

reduces to

$$I = \frac{\pi a^2 n r_D Z_1^2 Z_2^2}{3\theta} \frac{\partial F}{\partial \vec{p}} \cdot \left[\frac{\nabla \theta}{\theta} + \frac{\nabla n}{n} \right].$$

By definition, the integral $\int_{-\infty}^{\infty} d\vec{p}_2 F(2)$ is unity. To evaluate the other integral, we note that the interparticle potential, as given by Eq. (4.16) is

$$\phi_{12} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_o} \frac{1}{r},$$

so that

$$\frac{\partial \phi_{12}}{\partial \vec{q}_1} = - \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_o} \frac{1}{r^2} \frac{\partial r}{\partial \bar{q}_1} . \tag{III.2}$$

Now,
$$\vec{r} = \vec{q}_1 - \vec{q}_2$$

$$= (q_{1x}-q_{2x})\hat{i} + (q_{1y}-q_{2y})\hat{j} + (q_{1z}-q_{2z})\hat{k},$$

or,
$$r^2 = (q_{1x}-q_{2x})^2 + (q_{1y}-q_{2y})^2 + (q_{1z}-q_{2z})^2,$$

Now, from Appendix II we have

$$G(r, \bar{\theta}, \bar{n}) = [1 - \frac{aZ_1Z_2}{\theta_1 r} \exp(-\frac{r}{r_{D1}})]$$

$$+ \frac{aZ_1Z_2}{2\theta_1^2} [1 - \frac{r}{2r_{D1}}] \exp(-\frac{r}{r_{D1}}) \frac{\partial \theta}{\partial x}\Big|_1 \cdot \frac{x}{r}$$

$$+ \frac{aZ_1Z_2}{4\theta_1 n_1} \exp(-\frac{r}{r_{D1}}) \frac{\partial n}{\partial x}\Big|_1 \cdot \frac{x}{r_{D1}}$$

This can be written as

$$G(r, \bar{\theta}, \bar{n}) = G1 + G2 + G3,$$

where G1 is the first bracketed term, G2 and G3 the second and third terms, respectively.

Therefore,

$$I = - n \frac{\partial F(1)}{\partial \vec{p}_1} \frac{Z_1Z_2e^2}{4\pi\varepsilon_o} \cdot [\int_{-\infty}^{\infty} d\vec{q}_2 \frac{\vec{r}}{r^3} G1 + \int_{-\infty}^{\infty} d\vec{q}_2 \frac{\vec{r}}{r^3} G2$$

$$+ \int_{-\infty}^{\infty} d\vec{q}_2 \frac{\vec{r}}{r^3} G3]. \qquad \text{(III.4)}$$

The first integral of Eq. (III.4) is,

$$\int_{\infty}^{\infty} d\vec{q}_2 \frac{\vec{r}}{r^3} G1. \qquad \text{(III.5)}$$

Since $\vec{r} = \vec{q}_1 - \vec{q}_2$

$d\vec{r} = - d\vec{q}_2,$

$\vec{r} \to +\infty$ as $\vec{q}_2 \to -\infty,$

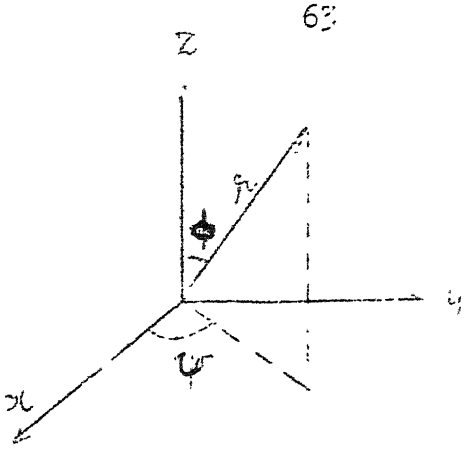

FIG III-1

so the expression (III.5) takes the form

$$\int_{-\infty}^{\infty} \frac{\vec{r}}{r^3} \left[1 - \frac{aZ_1Z_2}{r\theta_1} \exp\left(-\frac{r}{r_{D1}}\right)\right] d\vec{r}. \qquad \text{(III.6)}$$

Transforming to spherical coordinates as shown in Fig. III-1, we have,

$$\vec{r} = -[\hat{i}\, r \sin\phi \cos\psi + \hat{j}\, r \sin\phi \sin\psi + \hat{k} \cos\phi],$$

and $d\vec{r} = r^2 \sin\phi \, d\phi \, d\psi \, dr.$

In view of the above substitutions, the integral becomes,

$$-\int_0^{\infty} \hat{i} \left[1 - \frac{aZ_1Z_2}{r\theta_1} \exp\left(-\frac{r}{r_{D1}}\right)\right] dr \int_0^{\pi} \sin^2\phi \, d\phi \int_0^{2\pi} \cos\psi \, d\psi$$

$$-\int_0^{\infty} \hat{j} \left[1 - \frac{aZ_1Z_2}{r\theta_1} \exp\left(-\frac{r}{r_{D1}}\right)\right] dr \int_0^{\pi} \sin^2\phi \, d\phi \int_0^{2\pi} \sin\psi \, d\psi$$

$$-\int_0^{\infty} \hat{k} \left[1 - \frac{aZ_1Z_2}{r\theta_1} \exp\left(-\frac{r}{r_{D1}}\right)\right] dr \int_0^{\pi} \sin\phi\cos\phi \, d\phi \int_0^{2\pi} d\psi .$$

By direct integration it can be shown that,

$$\int_0^{2\pi} \cos\psi \, d\psi = 0, \quad \int_0^{2\pi} \sin\psi \, d\psi = 0 \text{ and } \int_0^{\pi} \sin\phi \cos\phi \, d\phi = 0.$$

Therefore, the first integral in Eq. (III.4), that is

$$\int_{-\infty}^{\infty} \frac{\vec{r}}{r^3} G1 \, d\vec{q}_2,$$

drops to zero.

The second integral is

$$\int_{-\infty}^{\infty} \frac{\vec{r}}{r^3} G2 \, d\vec{r} = -\frac{aZ_1Z_2}{2\theta_1^2} \frac{\partial\theta}{\partial x}\Big|_1 \int_{-\infty}^{\infty} \frac{\vec{r}}{r^3} \left[1 - \frac{r}{2r_{D1}}\right] \exp\left(-\frac{r}{r_{D1}}\right) \frac{x}{r} \, d\vec{r}$$

$$= -\frac{aZ_1Z_2}{2\theta_1^2} \frac{\partial\theta}{\partial x}\Big|_1 \left[\int_0^{\infty} \hat{i}\left(1 - \frac{r}{2r_{D1}}\right) \exp\left(-\frac{r}{r_{D1}}\right) dr \cdot \right.$$

$$\int_0^{\pi} \sin^3\phi \, d\phi \int_0^{2\pi} \cos\psi \, d\psi$$

$$+ \int_0^{\infty} \hat{j}\left(1 - \frac{r}{2r_{D1}}\right) \exp\left(-\frac{r}{r_{D1}}\right) dr \int_0^{\pi} \sin^3\phi d\phi \int_0^{2\pi} \sin\psi\cos\psi d\psi$$

$$\left. + \int_0^{\infty} \hat{k}\left(1 - \frac{r}{2r_{D1}}\right) \exp\left(-\frac{r}{r_{D1}}\right) dr \int_0^{2\pi} \sin^2\phi\cos\phi d\phi \int_0^{2\pi} \cos\psi d\psi \right]$$

(III.7)

Evaluating the integrals it can be shown that

$$\int_0^{2\pi} \sin\psi \cos\psi \, d\psi = 0 \quad \text{and} \quad \int_0^{2\pi} \cos\psi \, d\psi = 0$$

and to find $\int_0^{\pi} \sin^3\phi \, d\phi \int_0^{2\pi} \cos\psi \, d\psi$, we make use of the fact that

$$\sin^3\phi = \frac{3}{4}\sin\phi - \frac{1}{4}\sin 3\phi,$$

so that,

$$\int_0^{\pi} \sin^3\phi \, d\phi = \frac{4}{3},$$

and since

$$\int_0^{2\pi} \cos^2\psi \, d\psi = \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\psi) \, d\psi = \pi$$

We have,

$$\int_0^{\pi} \sin^3\phi \, d\phi \int_0^{2\pi} \cos^2\psi \, d\psi = \frac{4\pi}{3}.$$

With these results Eq. (III.7) takes the form

$$\int_{-\infty}^{\infty} \frac{\vec{r}}{r^3} G2 \, d\vec{r} = -\frac{aZ_1Z_2}{2\theta_1^2} \frac{\theta}{x}\Big|_1 \frac{4\pi}{3} \int_0^{\infty} \left(1 - \frac{r}{2r_{D1}}\right) \exp\left(-\frac{r}{r_{D1}}\right) dr$$

$$= \frac{2aZ_1Z_2\pi}{3\theta_1^2} \frac{\partial\theta}{\partial x}\Big|_1 \hat{i} \left[\int_0^\infty e^{-r/r_{D1}}\, dr - \frac{1}{2}\int_0^\infty \frac{r}{r_{D1}}\, e^{-r/r_{D1}}\, dr \right].$$

Let us put

$$\frac{r}{r_{D1}} \equiv s,$$

then the above equation takes the form

$$= -\frac{2aZ_1Z_2\pi}{3\theta_1^2} \frac{\partial\theta}{\partial x}\Big|_1 \hat{i} \left[r_{D1}\int_0^\infty e^{-s}\, ds - \frac{r_{D1}}{2}\int_0^\infty s\, e^{-s}\, ds \right]$$

$$= -\frac{2aZ_1Z_2\pi r_{D1}}{3\theta_1^2} \frac{\partial\theta}{\partial x}\Big|_1 \hat{i} \left[\Gamma(1) - \frac{1}{2}\Gamma(2) \right]$$

since $\Gamma(1) = \Gamma(2)$.

$$= -\frac{aZ_1Z_2r_{D1}\pi}{3\theta_1^2} \frac{\partial\theta}{\partial x}\Big|_1 \hat{i}.$$

Hence we have

$$\int_{-\infty}^{\infty} \frac{\vec{r}}{r^3}\, G_2\, d\vec{r} = -\hat{i}\, \frac{r_{D1}}{3}\, \frac{aZ_1Z_2\pi}{\theta_1^2}\, \frac{\partial\theta}{\partial x}\Big|_1 \qquad \text{(III.8)}$$

Finally we need to evaluate

$$\int_{-\infty}^{\infty} \frac{\vec{r}}{r^3}\, G_3\, d\vec{q}_2 = \int_{-\infty}^{\infty} \frac{\vec{r}}{r^3}\, \frac{aZ_1Z_2}{4\theta_1 n_1} \exp\left(-\frac{r}{r_{D1}}\right) \frac{\partial n}{\partial x}\Big|_1 \frac{x}{r_{D1}}\, d\vec{r}$$

$$= -\frac{aZ_1Z_2}{4\theta_1 n_1 r_{D1}} \frac{\partial n}{\partial x}\Big|_1 \Big[\hat{i} \int r \exp(-\frac{r}{r_{D1}}) dr \int_0^{\pi} \sin^3\phi d\phi \int_0^{2\pi} \cos^2\psi \, d\psi$$

$$+ \hat{j} \int_0^{\infty} r \exp(-\frac{r}{r_{D1}}) \, dr \int_0^{\pi} \sin^3\phi \, d\phi \int_0^{2\pi} \sin\psi\cos\psi \, d\psi$$

$$+ \hat{k} \int_0^{\infty} r \exp(-\frac{r}{r_{D1}}) \, dr \int_0^{\pi} \sin^2\phi \cos\phi \, d\phi \int_0^{2\pi} \cos\psi \, d\psi \Big].$$

The last two integrals vanish and we have

$$= -\frac{aZ_1Z_2\pi}{3\theta_1 n_1 r_{D1}} \frac{\partial n}{\partial x}\Big|_1 \hat{i} \Big[\int_0^{\infty} r \exp(-\frac{r}{r_{D1}}) \, dr\Big]$$

$$= -\hat{i} \frac{aZ_1Z_2\pi r_{D1}}{3\,\theta_1 n_1} \frac{\partial n}{\partial x}\Big|_1$$

So

$$\int_{-\infty}^{\infty} \frac{\vec{r}}{r^3} G_3 \, d\vec{r} = -\hat{i} \frac{aZ_1Z_2\pi r_{D1}}{3\,\theta_1 n_1} \frac{\partial n}{\partial x}\Big|_1 \qquad \text{(III.9)}$$

Putting Eqs. (III.8) and (III.9) into Eq. (III.4) we finally get

$$I = \frac{nZ_1Z_2e^2}{4\pi\varepsilon_o} \frac{\partial F(1)}{\partial \vec{p}_1} \cdot \Big[i \frac{r_{D1}}{3} \frac{aZ_1Z_2\pi}{\theta_1^2} \frac{\partial \theta}{\partial x}\Big|_1 + i \frac{r_{D1}}{3} \frac{aZ_1Z_2\pi}{\theta_1 n_1} \frac{\partial n}{\partial x}\Big|_1 \Big]$$

$$I = \frac{\pi a^2 Z_1^2 Z_2^2 n r_D}{3\theta} \frac{\partial F}{\partial v_p} \cdot \left[\frac{\nabla \theta}{\theta} + \frac{\nabla n}{n} \right] \qquad \text{(III.10)}$$

where we have dropped the subscript 1, and used the notation,

$$a \equiv \frac{e^2}{4\pi\varepsilon_o} \quad \text{(as defined in Appendix II).}$$

The right had number of Eq. (III.10) is the evaluated expression for the collision integral appearing in Eq. (4.17).

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